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2001 J. Phys. A: Math. Gen. 34 11185

(http://iopscience.iop.org/0305-4470/34/50/301)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 11185–11191 PII: S0305-4470(01)28124-4

Brownian parametric oscillator: analytical results for a high-frequency driving field

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Received 14 August 2001, in final form 29 October 2001
Published 7 December 2001
Online at stacks.iop.org/JPhysA/34/11185

Abstract

The dissipative dynamics of a classical parametric oscillator is studied analytically. For a generic functional form of the parametric driving, a simplified description of the system is obtained by performing a sequence of transformations set up from the deterministic Floquet solutions. In the high-frequency regime, the application of an averaging method leads to the description of the secular dynamics as an effective bidimensional Ornstein–Uhlenbeck process. The expressions obtained for the probability density and the correlation functions allow us to unravel the mechanisms responsible for the nontrivial dependence of the variances on the driving amplitude.

PACS number: 05.40.−a

The parametric oscillator, i.e., the harmonic oscillator with a periodic time-dependent frequency, is a fundamental model, which has played an important role in the study of diverse physical problems [1–8]. Extensive theoretical and experimental research on these systems has had important conceptual and practical implications. Indeed, multiple generalizations of the basic model, set up to describe increasingly complex systems, have allowed the analysis of the combined effect of parametric oscillations and other dynamical aspects such as nonlinearity, additional driving or dissipation (i.e., friction and noise in a classical regime or coupling to a reservoir in a quantum treatment). These studies, relevant to systems such as ions and electrons in traps [2–5], signals generated in electronic circuits [6] or cavity modes of the electromagnetic field [7], have uncovered a variety of dynamical behaviour, which, depending on the parameters, can show diverse fundamental effects. In particular, it is known that, in the regime of parametric resonance, both quantum and classical models can present squeezing [7, 8], i.e., a reduction in the fluctuations of one of the coordinates at the expense of an increase in the conjugate variance. The practical use of this property has led to significant advances in the implementation of noise reduction schemes [7, 8].

Recently, new results of conceptual and practical interest have been reported on this subject. For a trapped charged microparticle, modelled as a Brownian classical oscillator sinusoidally driven in frequency, experimental [2] and theoretical [1] research, carried out far from the parametric-resonance condition, has revealed a nontrivial dependence of the asymptotic variances on the driving amplitude. It has been found that, in certain ranges of parameters, as the amplitude grows, the position variance presents an initial reduction followed by a monotonic increase; whereas, the velocity variance increases monotonically. Remarkably, the magnitude of this effect, which can be termed *classical quadrature squeezing*, depends on the damping constant. It has also been found that the correlation functions present damped oscillations with a frequency that depends in a complex way on the driving and friction coefficients.

Our study focuses on these features and aims at unravelling the mechanisms responsible for them. In particular, we intend to investigate their possible general character. To this end, we leave the restrictions of sinusoidal driving and white noise, present in previous studies, and consider a generic periodic driving and a broadband noise. Then, working in a high-frequency regime, analytical expressions for the variances and correlation functions are obtained. In this framework, we give an understanding of some aspects of the peculiar stochastic dynamics, tracing them back to general properties of the deterministic system. Then, to illustrate our conclusions, the sinusoidal case is considered and some of the results of [1, 2] are recovered. Our methodology has two main sources. First, the changes of variables that lead to our simplified description are the classical counterparts of the unitary transformations presented in [3] (see also [4] for an alternative treatment) and applied in [5] to study the quantum dynamics of similar systems. Second, the coarse graining that allows analytical results to be obtained is based on the methods developed by Stratonovich to the averaging of stochastic processes [6].

We consider a time-dependent oscillator perturbed by a linear frictional force and an additive noise term. Specifically, we study the system defined by

$$\ddot{q} + k(t)q = -\gamma\dot{q} + (\gamma D)^{1/2}\,\xi(t) \tag{1}$$

with $k(t) = k(t+\tau)$. For a sinusoidal form of the driving field, namely, for $k(t) = \omega_0^2 + b\cos(\Omega t)$, where $\Omega \equiv 2\pi/\tau$; and, in the case of $\xi(t)$ being Gaussian white noise, equation (1) corresponds to the system studied in [1, 2].

We start by analysing the Hamiltonian dynamics, described by $\ddot{q} + k(t)q = 0$. This equation, which for a sinusoidal form of $k(t)$ is known as the Mathieu equation, has been extensively studied [9, 10]. Let us summarize here some of its well-known properties for a generic periodic driving. The Floquet theorem states that this second-order differential equation with periodic coefficients has solutions of the form $f(t) = \mathrm{e}^{\mathrm{i}\mu t}\phi(t)$, where $\phi(t)$ has the same periodicity as the coefficients, i.e., $\phi(t+\tau) = \phi(t)$, and $f(t)$ is bounded (*stable*) or unbounded (*unstable*) depending on the character, real or complex, of the *Floquet exponent* μ. As we are interested in a confined system, our study will be restricted to the functional forms of $k(t)$ that, irrespective of the initial conditions, lead to bounded solutions, which correspond to μ/Ω having a real noninteger value. In this case, the motion is quasiperiodic; $f(t)$ and its complex conjugate $f^*(t)$ form a pair of linearly independent solutions and, therefore, a basis to describe the dynamics. The stability and functional structure of $f(t)$ are not affected by a change in the phase of the driving, which, in fact, implies only a change in the time origin. Hence, without loss of generality, we continue our study assuming a phase equal to zero. Later on, the case of having the phase uniformly distributed between 0 and 2π, which can be relevant to experimental realizations of the model, will be considered.

Now we turn to the complete system. Given our interest in discussing the possible generality of the nontrivial stochastic features, we apply a methodology that will allow the analytical study to be completed without specifying the time dependence of $k(t)$. Parallelling the sequence of unitary transformations introduced in [3] to obtain the quasieigenstates of the quantum Hamiltonian counterpart, a simplified description of the complete system is built up by taking the following steps. (i) We make the change of variables $Q = q/|f|$, $P = |f|(p-2\chi q)$, where $p = \dot{q}$ and $\chi \equiv (\dot{f} f^* + \dot{f}^* f)/(4|f|^2)$. Note that a scale transformation is included in this first step and that $|f|$ and χ are both strictly periodic functions. (ii) We introduce the complex variable α, defined by the relation $\alpha = \sqrt{W/2}Q + \mathrm{i}\sqrt{1/2W}P$. α and its complex conjugate α^* are, respectively, the analogues of the annihilation and creation operators of the quantum study. W, which is given by the Wronskian of the two linearly independent quasiperiodic solutions as $2\mathrm{i}W = \dot{f} f^* - f\dot{f}^*$, is time independent; additionally, it is assumed that $W > 0$ [3]. The resulting equation for α reads

$$\dot{\alpha} = -\mathrm{i}\frac{W}{|f|^2}\alpha - \gamma\left(\frac{1}{2}(\alpha-\alpha^*) + \mathrm{i}\frac{\chi|f|^2}{W}(\alpha+\alpha^*)\right) + \mathrm{i}\frac{|f|}{\sqrt{2W}}\sqrt{\gamma D}\xi(t). \tag{2}$$

No approximations have been made up to this point. A necessary and sufficient condition for the validity of the applied treatment is the requirement of bounded motion for the Hamiltonian system. There are no additional restrictions on the specific time dependence of $k(t)$. Moreover, our method is valid irrespective of the spectral properties of the random force.

Despite its apparent complex structure, equation (2) provides a framework for implementing the averaging process in a straightforward way. Indeed, as the explicit time dependence of the parametric driving has been transferred to the terms that depend on χ and/or $|f|$, equation (2) corresponds to a Brownian harmonic oscillator with parameters modulated by the driving frequency and its harmonics. Therefore, if the magnitude of these parameters is much smaller than Ω, the driving period gives the smallest time scale in the problem and an average over τ ($\langle\cdots\rangle_\tau$) can be applied to account for the secular dynamics, which is consequently described in terms of a Brownian harmonic oscillator with effective time-independent parameters. Since the magnitude of the effective frequency is given by the Floquet exponent μ, the range of parameters where the coarse graining is valid is defined by $\mu, \gamma, \gamma\Omega/\mu, \gamma D/\mu \ll \Omega$; additionally, a sufficiently small correlation time t_{c} is required for the noise, specifically, $t_{\mathrm{c}} \ll 1/\mu$, [6, 11, 12]. These conditions define the high-frequency regime that will be assumed in the rest of our paper. In this regime, the averaging of both the deterministic part and the random terms in equation (2) is readily carried out. In particular, in the case of Gaussian white noise, i.e., for $\langle\xi(t)\rangle = 0$ and $\langle\xi(t)\xi(t')\rangle = 2\delta(t-t')$, the application of standard averaging methods (see [6], vol II, p 97) allows us to describe the secular evolution of the variables Q and P as

$$\begin{aligned}
\dot{Q} &= \left\langle\frac{1}{|f|^2}\right\rangle_\tau P \\
\dot{P} &= -\left(W^2\left\langle\frac{1}{|f|^2}\right\rangle_\tau + 2\gamma\langle\chi|f|^2\rangle_\tau\right)Q - \gamma P + \zeta(t)
\end{aligned} \tag{3}$$

which correspond to a bidimensional Ornstein–Uhlenbeck process [13], where the effective noise term is characterized by $\langle\zeta(t)\rangle = 0$ and $\langle\zeta(t)\zeta(t')\rangle = 2\langle|f|^2\rangle_\tau\gamma D\delta(t-t')$. Note that the simple form obtained for the moments of this averaged stochastic force is due to the white-noise character assumed for $\xi(t)$. If, instead, a broadband coloured noise is considered, the corresponding spectral density taken at Ω and at its harmonics appears explicitly in the coarse graining and determines the effective noise strength [6, 11, 12]. As the structure of the equations does not change when a finite correlation time is assumed, we conclude that the presence of broadband fluctuations does not qualitatively alter the dynamics. The robustness of the results of [1, 2] against variations in the noise correlation time is one of the findings of the present paper.

Before proceeding with our study, let us clarify some points of the previous derivation. First, note that, following [6], we use the term broadband noise to refer to fluctuations with correlation time much smaller than any other relevant time scale present in the emergent averaged system. Hence, our methodology cannot be applied to equations which include memory terms or fluctuations with diverging correlation times. Second, we emphasize the importance in our treatment of the transformations that lead to equation (2): they allow the application of the averaging process as they incorporate part of the effect of the driving field into the secular dynamics and transfer the explicit time dependence to the perturbative terms. The application of a coarse graining directly on equation (1) would imply working at a lower order of approximation; the validity of this procedure in a regime of very high frequency will be discussed later on. Third, the applicability of our scheme to quantum analogues has been emphasized by using a notation that reflects the existence of a parallellism with the quantum treatment. Indeed, from our results, some relevant features of the quantum dynamics can be predicted: the change in the main frequency, which is relevant to the precise definition of resonance, and the presence of squeezing far from the well-characterized regime of parametric resonance.

The 'stationary' probability density for the bidimensional Ornstein–Uhlenbeck process of equation (3) is readily obtained as [13]

$$w_{\mathrm{st}}(Q,P)=\left\langle\frac{1}{|f|^2}\right\rangle_\tau\frac{\omega_{\mathrm{ef}}}{2\pi D_{\mathrm{ef}}}\exp\left[-\frac{1}{2}\frac{\langle 1/|f|^2\rangle_\tau^2}{D_{\mathrm{ef}}}P^2-\frac{1}{2}\frac{\omega_{\mathrm{ef}}^2}{D_{\mathrm{ef}}}Q^2\right] \tag{4}$$

where the effective values of the frequency and diffusion constant are, respectively,

$$\omega_{\mathrm{ef}}^2=\left\langle\frac{1}{|f|^2}\right\rangle_\tau\left[W^2\left\langle\frac{1}{|f|^2}\right\rangle_\tau+2\gamma\langle\chi|f|^2\rangle_\tau\right] \tag{5}$$

$$D_{\mathrm{ef}}=D\langle|f|^2\rangle_\tau\left(\langle 1/|f|^2\rangle_\tau\right)^2. \tag{6}$$

The variances in the initial variables q and p can be obtained from the variances in Q and P by reversing the sequence of transformations. If one assumes that the phase of the external field is uniformly distributed, which is applicable to experimental realizations of the system where there is no possibility of fixing the phase, the consequent averaging over the driving period τ leads finally to the averaged asymptotic variances

$$\left\langle\sigma_{qq}^{\mathrm{as}}\right\rangle_\tau=\eta^2\frac{D}{\omega_{\mathrm{ef}}^2} \tag{7}$$

$$\left\langle\sigma_{pp}^{\mathrm{as}}\right\rangle_\tau=D+4\eta\langle\chi^2|f|^2\rangle_\tau\langle 1/|f|^2\rangle_\tau\frac{D}{\omega_{\mathrm{ef}}^2} \tag{8}$$

$$\left\langle\sigma_{qp}^{\mathrm{as}}\right\rangle_\tau=2\eta\langle\chi|f|^2\rangle_\tau\langle 1/|f|^2\rangle_\tau\frac{D}{\omega_{\mathrm{ef}}^2} \tag{9}$$

where $\eta=\langle|f|^2\rangle_\tau\langle 1/|f|^2\rangle_\tau$. As a proof of consistency, we remark that, in this scheme, the results for the undriven case are recovered by taking $\eta=1$ and $\chi(t)=0$; in particular, the crossed variance vanishes.

Let us discuss how our approach, although valid only in a particular range of parameters, gives insight into the general mechanisms underlying the stochastic dynamics. The functions $|f(t)|$, W and $\chi(t)$, encapsulate all the characteristics of the deterministic system which are relevant to the noisy process. In particular, some of their properties, which we summarize in the following, give the clues to understanding the emergence of squeezing. First, the quasiperiodic character assumed for the deterministic motion implies the departure of $|f(t)|$ from a constant value, and, in turn, the departure of η from 1. Second, W and, consequently, ω_{ef}, depends strongly on the Floquet exponent; moreover, this exponent gives a good approximation for the secular frequency near a purely periodic regime. Third, χ, which can be rewritten as $\mathrm{Re}\,(\phi\dot{\phi}^*)/(2|\phi|^2)$, is completely determined by the coefficients of the harmonics of Ω that contribute to $f(t)$; therefore it conveys information on the specific time features of the quasiperiodic trajectories. We emphasize that our scheme makes it clear how the change in the secular frequency and the quasiperiodic character of the solutions are rooted in the presence of the driving term. Note that it is precisely the time dependence of ϕ that reflects the quasiperiodicity of the Floquet solutions $f(t)$ (see the discussion of the limit of very high frequency that we present further on).

Applying these general arguments to the analysis of our results, some aspects of the stochastic dynamics are clarified. (a) The changes in the variances are differently rooted. Whereas for $\eta \simeq 1$, a mere change in the Floquet coefficient modifies ω_{ef}, and, therefore, alters $\langle\sigma_{qq}^{\mathrm{as}}\rangle_\tau$, a nonzero $\langle\chi^2|f|^2\rangle_\tau$ is needed, in the same regime, to vary the velocity variance. Furthermore, as the enhancement of $\langle\sigma_{pp}^{\mathrm{as}}\rangle_\tau$ depends on $\dot{\phi}$ and η, it becomes smaller as a periodic deterministic range is approached. (b) The dependence of W on μ along with the changes in η inside the stability region, can explain the qualitatively different behaviours detected in the position variance as the driving amplitude is increased. (c) Our results reflect also the shift caused by γ in the effective frequency (see equation (5)), and, consequently, in the variances, which is one of the main findings of [1]. We conjecture that, under conditions less restrictive than the ones in our study, in particular, for larger values of γ, the friction-increased stability of the solutions for the complete deterministic system (i.e., the system of equation (1) without the noise term), and the consequent dependence of the Floquet exponents on γ [1], can account for the importance of this cooperative action of friction and driving.

The correlation functions for our approximate system are straightforwardly obtained [13]. For instance, the averaged autocorrelation function of the position variable has the form

$$\langle\langle q(t)q(t')\rangle\rangle_\tau = \langle|f(t)||f(t')|\rangle_\tau \frac{D_{\mathrm{ef}}}{\omega_{\mathrm{ef}}^2} \frac{\lambda_1\,\mathrm{e}^{-\lambda_2(t'-t)} - \lambda_2\,\mathrm{e}^{-\lambda_1(t'-t)}}{\lambda_1 - \lambda_2} \tag{10}$$

where $\langle\langle\cdots\rangle\rangle_\tau$ denotes a double average, statistical and over the driving period; $\lambda_{1,2} = \frac{1}{2}\left[\gamma \pm \left(\gamma^2 - 4\omega_{\mathrm{ef}}^2\right)^{1/2}\right]$, and $t' \geqslant t$. Note that the asymptotic variance is consistently recovered for $t = t'$. This derivation clearly shows that the damped oscillations of the correlation functions, detected in the numerical study of the sinusoidal-driving case [1], are rooted in the Ornstein–Uhlenbeck characteristics of the secular dynamics. The robust character of these features is also evident, as we are still considering a generic functional form for the driving. Moreover, despite its restricted validity, the above expression allows us to guess the relevance of the relative magnitudes of γ and ω_{ef} to the appearance, in other regimes, of qualitatively different behaviours.

To illustrate the applicability of our methodology, let us consider a parametric driving with the functional form $k(t) = \omega_0^2 + b\,\cos(\Omega t)$, which corresponds to the case studied in [1] and [2]. The solutions of the Mathieu equation, which describe in this case the deterministic dynamics, are given in [10]. For small values of the amplitude, it is shown that, in first order in b/Ω^2 and in the limit $\omega_0^2/\Omega^2 \ll 1$, the following approximations can be made: $\mu^2 \simeq \omega_0^2 + b^2/(2\Omega^2)$, $W \simeq \mu$ and $\chi(t) \simeq -(b/(2\Omega|f|^2))\sin(\Omega t)$. The effective frequency of the system is therefore $\omega_{\mathrm{ef}} = \mu$ and the averaged asymptotic variances are

$$\langle\sigma_{qq}^{\mathrm{as}}\rangle_\tau \simeq \frac{D}{\omega_{\mathrm{ef}}^2} \simeq \frac{D}{\omega_0^2} - \frac{D}{2\omega_0^4\Omega^2}b^2 \tag{11}$$

$$\langle\sigma_{pp}^{\mathrm{as}}\rangle_\tau \simeq D + \frac{D}{2\omega_0^2\Omega^2}b^2 \tag{12}$$

$$\langle\sigma_{qp}^{\mathrm{as}}\rangle_\tau = 0. \tag{13}$$

For the chosen values of the amplitude b there is no effect of γ on ω_{ef}. As b increases, ω_{ef} becomes larger and the position variance diminishes, whereas the velocity variance increases. These results completely agree with those presented in [1, 2] for the same range of parameters. Despite the differences between the approaches used, the same functional dependence and numerical factors are found. In particular, in the limit $\omega_0 \to 0$, which is the case studied in [2], we find $\omega_{\mathrm{ef}} \sim b^2/(2\Omega^2)$, and, therefore, $\langle\sigma_{qq}^{\mathrm{as}}\rangle_\tau \simeq 2D\Omega^2/b^2$. Of course one cannot observe here the increase and ultimate divergence of $\langle\sigma_{qq}^{\mathrm{as}}\rangle_\tau$ detected in [1] for larger driving amplitudes. Nevertheless, from the understanding given by our approach, one can reasonably conjecture that the change in the stability of the deterministic solutions must be crucial for the appearance of those effects. Also, it is observed that, in the considered regime of bounded deterministic motion and small friction, the correlation functions present, in agreement with the findings of [1], exponentially damped oscillations; the exponent is $\gamma/2$; the frequency, given by $\frac{1}{2}\left(4\omega_{\mathrm{ef}}^2 - \gamma^2\right)^{1/2}$, shows a dependence on the driving and damping parameters that reflects the changes in the periodicity detected numerically [1].

From the above picture one dynamical feature stands out: in the undriven case, because of the thermal equilibrium condition, the variances of the normalized variables take their thermal values, i.e., $\omega_0^2\sigma_{qq}^{\mathrm{as}} = D$ and $\sigma_{pp}^{\mathrm{as}} = D$ $(D = k_B T)$; the external field, which drives the system out of equilibrium, changes them in a nontrivial way. In particular, it beats the thermal limit for $\langle\sigma_{qq}^{\mathrm{as}}\rangle_\tau$, as we have found, mainly because of an increase in the effective frequency. We stress that, since the product of the averaged variances retains, in our first-order approximation, its *standard* value, $\omega_0^2\langle\sigma_{qq}^{\mathrm{as}}\rangle_\tau\langle\sigma_{pp}^{\mathrm{as}}\rangle_\tau = D^2 + O(b^2/\Omega^4)$, we can term the behaviour found classical quadrature squeezing [8] and consider the present model system, because of its simplicity, as a suitable scenario for implementing an effective noise reduction. An additional remark is in order: instead of the exponential reduction of the variance achieved in standard squeezing schemes, in our system, the reduction is quadratic in the field amplitude.

Finally, let us briefly present some results valid in the regime of very high frequency, as they clarify our findings. In this limit, the Kapitsa–Landau method [11, 14] can be applied, and, neglecting the high-frequency oscillations from the periodic secular motion, the system, which is always stable, is equivalent to an oscillator with a larger effective frequency, namely, $\omega_{\mathrm{ef}}^2 \simeq \omega_0^2 + b^2/(2\Omega^2)$. Hence, the position variance decreases monotonically with the driving amplitude, without experiencing any subsequent increase. This result gives plausibility to our previous conjecture on the link between a change in the stability of the system and the increase in the position variance. Moreover, the velocity variance does not change in this limit, which is also explained in our framework: as in this case the solutions are strictly periodic, $\phi(t)$ takes a constant value, then, $\chi(t) = 0$, and, consequently (see equation (8)), $\langle\sigma_{pp}^{\mathrm{as}}\rangle_\tau = D$.

It is worth making some final remarks. First, our analytical results for the Brownian parametric oscillator identify the deterministic roots of the reduction and the enhancement induced in the variances by the driving field. Indeed, they reveal the generality of these effects and how the magnitude of the changes depends on functions entirely given by the deterministic Floquet solutions. As a consequence, the possibility of designing strategies of control is open. Second, for the particular case of sinusoidal driving we have linked the decrease in $\langle\sigma_{qq}^{\mathrm{as}}\rangle_\tau$ as the driving amplitude grows, with the increase in the effective frequency of the oscillator, detected in our description. Furthermore, the expression obtained for ω_{ef} explains the changes in the periodicity of the correlation function [1]. Third, the dependence of $\langle\sigma_{pp}^{\mathrm{as}}\rangle_\tau$ on the driving amplitude, which is not explained by an increase in the effective frequency, has its origin also in the deterministic dynamics: in this case it is the function $\chi^2(t)$, which contains information on the particular time features of the Floquet solutions, that gives the magnitude of this effect. Finally, we remark on the potential practical implications of the study, given the robust character of the effects analysed and the generality of the model system considered.

Acknowledgment

One of the authors (SB) acknowledges financial support from the Basque Government (grant PI 1999-28).

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